

Background Scale Independence in Quantum Gravity

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Abstract

We study the background scale independence in single-metric approximation to the functional renormalization group equation (FRGE) for quantum gravity and show that it is possible to formulate it without using higher-derivative gauge fixing in arbitrary dimensions if we adopt the Landau gauge and suitable cutoff scheme. We discuss this problem for both the linear and exponential splits of the metric into background and fluctuations. The obtained modified Ward identity for the global rescalings of the background metric can be combined with the FRGE to give a manifestly scale-invariant solution. An explicit example of the FRGE is given for four-dimensional $f(R)$ gravity in this framework.

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1 Introduction

Asymptotic safety is one of the promising approaches to the formulation of quantum gravity within the framework of conventional field theory [1]. This program considers the possibility of having an interacting quantum field theory of gravitation, originated from a non-Gaussian ultraviolet (UV) fixed point in the theory space, in the nonperturbative renormalization group (RG) framework [2, 3, 4]. The existence of such a fixed point (scaling solution) would permit having an RG trajectory in the theory space flowing to it and characterized by all the dimensionless couplings remaining finite when the UV cutoff is removed. If the number of relevant operators is finite, this theory has predictive power. The approach has produced a wealth of results. For reviews and introductions, see [5, 6, 7].

To actually pursue this line, one has to make approximation such as truncation, derivative expansion etc. Most approaches keep a finite number of local operators in the effective action. These polynomial truncations may be viewed as based on a small curvature expansion and have arrived at amazingly higher-order expansion up to 34th order in the scalar curvature [8]. However, this still might not be good enough to draw the conclusion that asymptotic safety is achieved convincingly as long as one retains a finite number of operators. To go beyond this, one has to keep an infinite number of operators, which enables one to treat them without expansion. In the so-called $f(R)$ approximation [9]-[19], a Lagrangian of the form $f(R)$, not just a polynomial expansion, is used in which all possible forms in the scalar curvature are considered.

In this approach, a momentum cutoff k is introduced. The quantum effective action is recovered when we take the $k \rightarrow 0$ limit. It has been pointed out that a problem arises in a large-curvature regime where k is smaller than the minimum eigenvalues of the Laplacian [20, 18, 19]. If the spectrum of all operators has a finite gap δ , then for $k < \delta$ the flow equation does not integrate out any modes. This raises the question of the meaning of coarse-graining on length scales that are larger than the size of the manifold.

Recently, Morris has pointed out that this problem arises because the background independence is not respected in the single-metric formulation adapted in these approaches [21]. Background independence means that physics should not depend on the choice of the background. There are several approaches to address this issue. One is to use bimetric truncations [22, 23] and require shift invariance. Another is to solve the modified Ward identity (mWI) for the shift transformation and the flow equation simultaneously [24, 25, 26].

Shift invariance is a symmetry that keeps the classical action invariant. If one uses a linear split of the metric into background \bar{g} and fluctuation h :

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (1.1)$$

the classical action is invariant under a simultaneous shift of the background and fluctuation:

$$\delta\bar{g}_{\mu\nu} = \epsilon_{\mu\nu}, \quad \delta h_{\mu\nu} = -\epsilon_{\mu\nu}. \quad (1.2)$$

This should be a symmetry of the effective action, but is broken by gauge fixing and cutoff terms in the process of quantization and regularization. The problem associated with this was pointed out in [27].

Morris [21] has restricted the shift transformation to constant rescaling as a first step towards realizing the full invariance under shift symmetry and discussed the above problem associated with the coarse-graining. The basic idea is that it is wrong to relate the cutoff scale k to the fixed background; rather all background metrics of different overall scales should be treated

equally. He has been able to derive the mWI corresponding to the rescaling and show that it is compatible with the functional renormalization group equation (FRGE). It is found that the resulting solution to the FRGE is written in terms of scale-invariant variables. The background metric is no longer just a fixed one but is replaced by a dynamical variable, and we have to consider that it describes ensemble of metrics of different scales. In this way theories on a continuous infinity of manifolds are incorporated and the above problem related to the coarse-graining may be resolved. Unfortunately, he was able to do this only for six dimensions.

More recently it has been pointed out that the formulation can be extended to arbitrary dimensions if one uses 1) the exponential split of the metric into background and fluctuation, 2) higher-derivative gauge fixing and 3) special cutoff scheme [28]. The exponential split of the metric is defined by [29]

$$g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho_\nu. \quad (1.3)$$

It would be an interesting problem to study if the above three conditions are absolutely necessary.

In this paper we point out that Morris's formulation with the usual gauge fixing may be extended to arbitrary dimensions if one uses the Landau gauge and a cutoff scheme similar to but slightly different from that of Percacci and Vacca [28]. We show why the Landau gauge is singled out in order to make this formulation work and emphasize that it is not necessary to adopt exponential split. Since the exponential parametrization (1.3) has been shown to have various virtues compared to the linear split [17, 18, 19, 30, 31, 32], even though the consistent mWI can be derived in arbitrary dimensions with the linear split, it is interesting to see if the result may also be extended to the exponential split in arbitrary dimensions. We thus discuss this approach in both linear and exponential splits of the metric.

This paper is organized as follows. In the next section, we set off to discuss how to implement the rescaling invariance with the linear split in arbitrary dimensions. We first define the global rescaling transformation and show how the invariance of the gauge-fixing term forces us to choose the Landau gauge if we use the ordinary gauge-fixing condition. This is also noted by Morris, but the use of the auxiliary field makes the discussion more transparent. Then, motivated by Percacci and Vacca [28], we adopt what is called a “pure” cutoff scheme [33] that breaks the invariance but in a way that is compatible with the FRGE in arbitrary dimensions. In this process we have to modify the ghost part slightly compared to [28] because our gauge fixing is the usual one without higher derivatives. In this way we derive the mWI and arrive at a solution that is compatible with scale invariance. In sect. 3, we go on to discuss the same problem with the exponential split. We obtain basically the same result with usual gauge fixing but with a suitably modified pure cutoff in arbitrary dimensions. In sect. 4, we discuss how the FRGE looks with our pure cutoff, but the result is rather complicated. We leave discussions of the solution for future study. Section 5 is devoted to conclusions.

2 Modified Ward identity in the linear split

In this section, we derive the mWI in the linear split (1.1) valid for arbitrary dimensions, and show its compatibility with FRGE.

In either parametrization (1.1) or (1.3), the partition function is given by

$$Z[g_{\mu\nu}, J^{\alpha\beta}] = \int [Dh_{\mu\nu}] \exp \left[-S_0[g_{\mu\nu}, h_{\mu\nu}] + \int J^{\mu\nu} h_{\mu\nu} \right]. \quad (2.1)$$

Throughout this paper, the indices are raised or lowered by the background metric \bar{g} unless

otherwise stated. We consider d -dimensional Euclidean spacetime. By \int , we mean $\int d^d x$ and the \sqrt{g} factor is absorbed into the definition of $J^{\mu\nu}$.

2.1 Rescaling transformation

We would like to impose background scale independence under the constant rescaling

$$\delta \bar{g}_{\mu\nu} = 2\epsilon \bar{g}_{\mu\nu}, \quad (2.2)$$

together with

$$\delta h_{\mu\nu} = -2\epsilon \bar{g}_{\mu\nu}, \quad (2.3)$$

so that the total metric (1.1) does not change. We decompose the fluctuation field into traceless and trace parts:

$$h_{\mu\nu} = h_{\mu\nu}^T + \frac{1}{d} \bar{g}_{\mu\nu} h. \quad (2.4)$$

The variation gives

$$\delta h_{\mu\nu} = \delta h_{\mu\nu}^T + \frac{1}{d} \delta \bar{g}_{\mu\nu} h + \frac{1}{d} \bar{g}_{\mu\nu} \delta h. \quad (2.5)$$

Using (2.2) and (2.3), and taking the trace, we find

$$\delta h = -2\epsilon(h + d). \quad (2.6)$$

Substituting (2.6) back into (2.5), we get

$$\delta h_{\mu\nu}^T = 0. \quad (2.7)$$

The gauge-fixing and Faddeev-Popov (FP) terms may be obtained most easily by using the BRST transformation. This can be obtained by replacing the parameters of the coordinate transformation by the ghost:

$$\delta_B g_{\mu\nu} = -\delta\lambda (g_{\mu\alpha} \nabla_\nu C^\alpha + g_{\nu\alpha} \nabla_\mu C^\alpha), \quad (2.8)$$

where $\delta\lambda$ is an anticommuting parameter. This should be regarded as the transformation of the fluctuation field $h_{\mu\nu}$ since the quantum gauge transformation is generated when $\bar{g}_{\mu\nu}$ is held fixed.

$$\delta_B h_{\mu\nu} = -\delta\lambda (g_{\mu\alpha} \nabla_\nu C^\alpha + g_{\nu\alpha} \nabla_\mu C^\alpha), \quad (2.9)$$

We should note that here the covariant derivative is defined by the whole metric. The BRST transformation for other fields is derived by the requirement of the nilpotency of the transformation:

$$\delta_B C^\mu = \delta\lambda C^\rho \partial_\rho C^\mu, \quad \delta_B \bar{C}_\mu = i\delta\lambda B_\mu, \quad \delta_B B_\mu = 0, \quad (2.10)$$

where \bar{C}_μ is the FP anti-ghost and B_μ is an auxiliary field that enforces the gauge-fixing condition. The gauge-fixing function is

$$F_\mu = \bar{g}^{\rho\nu} \left(\bar{\nabla}_\rho h_{\mu\nu} - \frac{b+1}{d} \bar{\nabla}_\mu h_{\rho\nu} \right), \quad (2.11)$$

where b is a gauge parameter. Here and in what follows, the bar on the covariant derivative means that it is constructed with the background metric. Note that $\bar{\nabla}_\mu$ is invariant under the rescaling transformation. The gauge-fixing and FP terms are then given as [32]

$$\begin{aligned}\mathcal{L}_{GF+FP}/\sqrt{\bar{g}} &= i\delta_B \left[\bar{g}^{\mu\nu} \bar{C}_\mu \left(F_\nu + \frac{a}{2} B_\nu \right) \right] / \delta\lambda \\ &= -\bar{g}^{\mu\nu} B_\mu \left(F_\nu + \frac{a}{2} B_\nu \right) - i\bar{C}_\mu \Delta^{(gh)\mu}{}_\nu C^\nu,\end{aligned}\quad (2.12)$$

where a is another gauge parameter and

$$\Delta^{(gh)\mu}{}_\nu \equiv -\bar{g}^{\mu\lambda} \bar{g}^{\rho\sigma} \left(\bar{\nabla}_\rho (g_{\lambda\nu} \nabla_\sigma + g_{\sigma\nu} \nabla_\lambda) - \frac{b+1}{d} \bar{\nabla}_\lambda (g_{\rho\nu} \nabla_\sigma + g_{\sigma\nu} \nabla_\rho) \right). \quad (2.13)$$

Since the B_μ field involves no derivatives, if $a \neq 0$, we can simply integrate it out and we are left with the gauge-fixing and FP ghost terms. However, this is not convenient for our purpose.

Under the rescaling transformation (2.2), we have

$$\delta\sqrt{\bar{g}} = d\epsilon\sqrt{\bar{g}}, \quad (2.14)$$

where $\bar{g} = \det(\bar{g}_{\mu\nu})$. If we assume that the auxiliary field B_μ has dimension d_B , the quadratic term in the auxiliary field transforms like

$$\delta \left(-\frac{a}{2} \sqrt{\bar{g}} \bar{g}^{\mu\nu} B_\mu B_\nu \right) = -\frac{a}{2} (d-2+2d_B) \epsilon \sqrt{\bar{g}} \bar{g}^{\mu\nu} B_\mu B_\nu, \quad (2.15)$$

so this is invariant if we take

$$d_B = \frac{2-d}{2}, \quad \text{or} \quad \delta B_\mu = \frac{2-d}{2} \epsilon B_\mu. \quad (2.16)$$

The rescaling transformation of the gauge-fixing function is

$$\delta F_\mu = -2\epsilon F_\mu. \quad (2.17)$$

This leads to

$$\delta(\sqrt{\bar{g}} \bar{g}^{\mu\nu} B_\mu F_\nu) = \frac{d-6}{2} (\sqrt{\bar{g}} \bar{g}^{\mu\nu} B_\mu F_\nu), \quad (2.18)$$

so that this is invariant only for $d=6$, as found by Morris [21].

Is there no way to make it invariant for any dimension? Actually if we take the Landau gauge $a=0$, we do not have to take the rescaling dimension of the auxiliary field B_μ as (2.16). We can just require that (2.18) is invariant to find

$$\delta B_\mu = (4-d)\epsilon B_\mu. \quad (2.19)$$

Note that it is conceptually better to keep B_μ rather than to eliminate it for $a=0$. According to the BRST symmetry (2.9) and (2.10), we should assign the rescaling dimension to each field as

$$\delta C^\mu = 0, \quad \delta \bar{C}_\mu = (4-d)\epsilon \bar{C}_\mu. \quad (2.20)$$

Note that C^μ and \bar{C}_μ are independent Hermitian fields, so they can have different dimensions. It is then easy to see that the FP ghost term is also scale invariant. We thus see that the Landau gauge is necessarily singled out by the rescaling invariance of these terms in this formulation. We

note that Ref. [28] achieved the invariance by introducing higher-derivative gauge fixing without dimensionful parameters, and in that case we do not have to take the Landau gauge. We thus confirm here in slightly improved way that higher-derivative gauge fixing is not necessary if we choose the Landau gauge, as discussed in [21].

In the usual gauge $a \neq 0$, we have a contribution from the square term of the gauge-fixing function. In the Landau gauge, there is also effectively the same contribution. When we integrate over the auxiliary field B_μ , this produces δ -function, which must be path integrated by the fields. Then we have to take into account the Jacobian from the gauge-fixing function (2.11). Because the Landau gauge strongly imposes the gauge condition to be zero, the mode appearing there does not appear in other parts of the action. Denoting this mode by ρ_μ , it has a similar transformation property to the ghost term, and we can make it scale invariant by assigning suitable rescaling dimension to that. We will also use the same cutoff for this term as the FP ghost term. In this case we can see that the contribution almost cancels against the ghost term. Since the discussion is basically the same as for the ghost, we suppress this for the moment, and discuss this in more detail in sect 3.2. A related discussion is given in [14].

Next come the cutoff terms that break the rescaling invariance. Morris considered a cutoff that is related to the Hessians of the kinetic terms. As a result, the consistency of the FRGE and the modified scale identity again requires that the spacetime dimension be six. However it was pointed out in [28] that this is not necessary if we use suitable cutoff. Here we have to further modify that because the rescaling dimensions are different. We thus consider

$$\begin{aligned}\Delta S_k(h_{\mu\nu}^T, \bar{g}_{\mu\nu}) &= \frac{1}{2} \int \sqrt{\bar{g}} [h_{\mu\nu}^T \bar{g}^{\mu\rho} \bar{g}^{\nu\rho} R_k^T(\bar{\Delta}) h_{\rho\sigma}^T + h R_k(\bar{\Delta}) h], \\ \Delta S_k^{gh}(\bar{C}_\mu, C^\mu, \bar{g}_{\mu\nu}) &= -i \int \sqrt{\bar{g}} \bar{C}_\mu R_k^{gh}(\bar{\Delta}) C^\mu,\end{aligned}\tag{2.21}$$

where we choose

$$R_k^T(\bar{\Delta}) = ck^{d-4}r(y), \quad R_k(\bar{\Delta}) = c_0k^{d-4}r(y), \quad R_k^{gh}(\bar{\Delta}) = c_{gh}k^4r(y),\tag{2.22}$$

with $y = \bar{\Delta}/k^2$ and suitable coefficients c, c_0 , and c_{gh} . Here, r is a dimensionless function that vanishes rapidly for $y > 1$ and $r(0) = 1$. These are independent of any parameters in the action, and called a “pure” cutoff [33]. Note that the power of the cutoff k is different from those in [28]. This is necessary in order to achieve the nice transformation property of these terms. Denoting $t = \ln k$, we then have

$$\delta R_k^T = \epsilon[\partial_t R_k^T - (d-4)R_k^T], \quad \delta R_k = \epsilon[\partial_t R_k - (d-4)R_k], \quad \delta R_k^{gh} = \epsilon[\partial_t R_k^T - 4R_k^{gh}],\tag{2.23}$$

for the above cutoffs, and

$$\begin{aligned}\delta \Delta S_k &= \frac{\epsilon}{2} \int \sqrt{\bar{g}} [h_{\mu\nu}^T \bar{g}^{\mu\rho} \bar{g}^{\nu\rho} \partial_t R_k^T(\bar{\Delta}) h_{\rho\sigma}^T + h \partial_t R_k(\bar{\Delta}) h - 4d R_k(\bar{\Delta}) h], \\ \delta \Delta S_k^{gh} &= -i\epsilon \int \sqrt{\bar{g}} \bar{C}_\mu \partial_t R_k^{gh}(\bar{\Delta}) C^\mu.\end{aligned}\tag{2.24}$$

Note that there is no term proportional to ΔS_k which was present in [21] with a factor $(d-6)$.

2.2 Modified Ward identity

The generating functional of the Green functions are given by

$$e^{W_k[\bar{g}, J]} = \int [\mathcal{D}g] e^{-S[g] - \Delta S_k[\bar{g}, h] + \int (J^{\mu\nu} h_{\mu\nu}^T + J_h h)},\tag{2.25}$$

where we have suppressed tensor indices. Since W_k is a functional of $\bar{g}_{\mu\nu}$ and J , it does not have transformation under the variation of $h_{\mu\nu}$. Therefore under the rescaling transformation with J fixed, we get

$$\int \frac{\delta W_k}{\delta \bar{g}_{\mu\nu}} \delta \bar{g}_{\mu\nu} = -\langle \delta \Delta_k S \rangle + \int J^{\mu\nu} \langle \delta h_{\mu\nu}^T \rangle + \int J_h \langle \delta h \rangle. \quad (2.26)$$

We now define

$$\Gamma_k = -W_k + \int J^{\mu\nu} \langle h_{\mu\nu}^T \rangle + \int J_h \langle h \rangle - \Delta S_k[\bar{g}_{\mu\nu}, \langle h_{\mu\nu}^T \rangle, \langle h \rangle], \quad (2.27)$$

where $\langle h \rangle$ stands for the expectation values of $h_{\mu\nu}^T$ and h . We then note that

$$\begin{aligned} \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} &= -\frac{\delta W_k}{\delta \bar{g}_{\mu\nu}} - \frac{\delta \Delta S_k[\bar{g}, \langle h_{\mu\nu}^T \rangle, \langle h \rangle]}{\delta \bar{g}_{\mu\nu}}, \\ \frac{\delta \Gamma_k}{\delta \langle h_{\mu\nu}^T \rangle} &= J^{\mu\nu} - \frac{\delta \Delta S_k[\bar{g}, \langle h_{\mu\nu}^T \rangle, \langle h \rangle]}{\delta \langle h_{\mu\nu}^T \rangle}, \\ \frac{\delta \Gamma_k}{\delta \langle h \rangle} &= J_h - \frac{\delta \Delta S_k[\bar{g}, \langle h_{\mu\nu}^T \rangle, \langle h \rangle]}{\delta \langle h \rangle}, \\ \frac{\delta W_k}{\delta J^{\mu\nu}} &= \langle h_{\mu\nu}^T \rangle, \quad \frac{\delta W_k}{\delta J_h} = \langle h \rangle. \end{aligned} \quad (2.28)$$

Using these in Eq. (2.26), we find [34]

$$\int \left(\frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} + \frac{\delta \Delta S_k}{\delta \bar{g}_{\mu\nu}} \right) \delta \bar{g}_{\mu\nu} = \langle \delta \Delta S_k \rangle - \int \left(\frac{\delta \Gamma_k}{\delta \langle h_{\mu\nu}^T \rangle} + \frac{\delta \Delta S_k}{\delta \langle h_{\mu\nu}^T \rangle} \right) \langle \delta h_{\mu\nu}^T \rangle - \int \left(\frac{\delta \Gamma_k}{\delta \langle h \rangle} + \frac{\delta \Delta S_k}{\delta \langle h \rangle} \right) \langle \delta h \rangle. \quad (2.29)$$

Together with (2.24), this leads to

$$\begin{aligned} \int \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} \delta \bar{g}_{\mu\nu} + \int \frac{\delta \Gamma_k}{\delta \langle h \rangle} \langle \delta h \rangle &= \frac{\epsilon}{2} \int \sqrt{\bar{g}} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} \partial_t R_k^T \frac{\delta^2 W_k}{\delta J_{\mu\nu} \delta J_{\rho\sigma}} + \frac{\epsilon}{2} \int \sqrt{\bar{g}} \partial_t R_k \frac{\delta^2 W_k}{\delta J_h \delta J_h} \\ &\quad - i\epsilon \int \sqrt{\bar{g}} \bar{C}_\mu \partial_t R_k C^\mu. \end{aligned} \quad (2.30)$$

In this way we finally arrive at the mWI

$$\begin{aligned} &\epsilon \left[2 \int \bar{g}_{\mu\nu} \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} - 2d \int \frac{\delta \Gamma_k}{\delta \langle h \rangle} - 2 \int \langle h \rangle \frac{\delta \Gamma_k}{\delta \langle h \rangle} \right] \\ &= \epsilon \left[\frac{1}{2} \text{Tr} \left\{ \left(\frac{\delta^2 \Gamma_k}{\delta h^T \delta h^T} + R_k^T \right)^{-1} \partial_t R_k^T \right\} + \frac{1}{2} \text{Tr} \left\{ \left(\frac{\delta^2 \Gamma_k}{\delta h \delta h} + R_k \right)^{-1} \partial_t R_k \right\} \right. \\ &\quad \left. - \text{Tr} \left\{ \left(\frac{\delta^2 \Gamma_k}{\delta \bar{C} \delta C} + R_k^{gh} \right)^{-1} \partial_t R_k^{gh} \right\} \right]. \end{aligned} \quad (2.31)$$

Apart from the factor ϵ , the RHS is identical to the RHS of the exact RG equation. We thus get

$$\int \left[2 \bar{g}_{\mu\nu} \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} - 2d \frac{\delta \Gamma_k}{\delta \langle h \rangle} - 2 \langle h \rangle \frac{\delta \Gamma_k}{\delta \langle h \rangle} \right] - k \frac{d\Gamma_k}{dk} = 0. \quad (2.32)$$

2.3 Single-metric approximation

We now make the standard single-metric approximation [4]. In this approximation, we keep only the dependence on the constant part \bar{h} defined by

$$\bar{h} = \frac{1}{V} \int \sqrt{\bar{g}} h, \quad h^\perp = h - \frac{1}{V} \int \sqrt{\bar{g}} h, \quad (2.33)$$

where $V = \int \sqrt{\bar{g}}$. As discussed in detail by Morris [21], we have

$$\begin{aligned} \int \frac{\partial \Gamma}{\partial h} &= \frac{\partial \Gamma}{\partial \bar{h}}, \\ \int h \frac{\partial \Gamma}{\partial h} &= \bar{h} \frac{\partial \Gamma}{\partial \bar{h}} + \int h^\perp \frac{\partial \Gamma}{\partial h^\perp}. \end{aligned} \quad (2.34)$$

The last term in the second equation is discarded in our approximation. Substituting these into (2.32), we get

$$2 \int \bar{g}_{\mu\nu} \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} - 2d \frac{\delta \Gamma_k}{\delta \bar{h}} - 2\bar{h} \frac{\delta \Gamma_k}{\delta \bar{h}} - k \frac{d\Gamma_k}{dk} = 0. \quad (2.35)$$

The solution is given by

$$\Gamma = \hat{\Gamma}_{\hat{k}}[\hat{g}_{\mu\nu}], \quad (2.36)$$

where $\hat{g}_{\mu\nu}$ is defined as

$$\hat{g}_{\mu\nu} = \left(1 + \frac{\bar{h}}{d}\right) \bar{g}_{\mu\nu}, \quad (2.37)$$

and

$$\hat{k} = k / \sqrt{1 + \bar{h}/d}. \quad (2.38)$$

Thus we arrive at the same flow equation in the single-metric approximation with the background metric $\bar{g}_{\mu\nu}$ and cutoff scale k replaced by $\hat{g}_{\mu\nu}$ and \hat{k} , respectively, in arbitrary dimensions. The metric $\hat{g}_{\mu\nu}$ is no longer a fixed background but becomes dynamical through dependence on \bar{h} . We should consider that the solution describes an ensemble of different scales. More precisely, it describes an infinite ensemble of background spacetimes related by the rescaling $\bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}/\alpha^2$, which is compensated by $\bar{h} \rightarrow (\bar{h} + d)\alpha^2 - d$ and $k \rightarrow k\alpha$. The variables $\hat{g}_{\mu\nu}$ and \hat{k} are scale independent.

In this way we have been able to extend the scale-invariant formulation to arbitrary dimensions without using higher-derivative gauge-fixing or exponential parametrization. What is important is to adopt a suitable cutoff and Landau gauge, which enable us to make the gauge-fixing and FP ghost terms rescaling invariant.

3 Modified Ward identity in the exponential split

In a practical discussion of the FRGE, it is quite often useful to use the exponential parametrization. In this section, we derive the mWI in the exponential split (1.3) without higher-derivative gauge fixing. The main difference from the linear split will be in the rescaling dimension of various terms.

3.1 Scale transformation and the derivation

Let us require background rescaling independence under

$$\delta\bar{g}_{\mu\nu} = 2\epsilon\bar{g}_{\mu\nu}, \quad (3.1)$$

together with

$$\delta h_{\mu\nu} = 2\epsilon(h_{\mu\nu} - \bar{g}_{\mu\nu}), \quad (3.2)$$

so that the total metric (1.3) does not change [28]. Here we also decompose the fluctuation field into traceless and trace parts just as in (2.4). Using (3.1) and (3.2), and taking the trace, we find

$$\delta h_{\mu\nu}^T = 2\epsilon h_{\mu\nu}^T, \quad \delta h = -2d\epsilon, \quad (3.3)$$

in contrast to (2.7).

The gauge-fixing and FP terms may be obtained as in the preceding section. The BRST transformation of the fluctuation $h_{\mu\nu}$ is complicated and we adopt the result in [28]. Denoting the metric split (1.3) as

$$\mathbf{g} = \bar{\mathbf{g}}e^{\mathbf{X}}, \quad \mathbf{X} = \bar{\mathbf{g}}^{-1}\mathbf{h}, \quad (3.4)$$

in a matrix notation, the BRST transformation corresponding to reparametrization is given by

$$\delta_B \mathbf{X} = \delta\lambda \frac{ad_{\mathbf{X}}}{e^{ad_{\mathbf{X}}} - 1} (\bar{\mathbf{g}}^{-1} \mathcal{L}_C \bar{\mathbf{g}} + \mathcal{L}_C e^{\mathbf{X}} e^{-\mathbf{X}}), \quad (3.5)$$

where $ad_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$ and

$$\mathcal{L}_C g_{\mu\nu} = g_{\rho\nu} \nabla_\mu C^\rho + g_{\rho\mu} \nabla_\nu C^\rho. \quad (3.6)$$

We use the same gauge-fixing function (2.11) as in the previous section. The gauge-fixing and FP terms are then given as [32]

$$\begin{aligned} \mathcal{L}_{GF+FP}/\sqrt{\bar{g}} &= i\delta_B \left[\bar{g}^{\mu\nu} \bar{C}_\mu \left(F_\nu + \frac{a}{2} B_\nu \right) \right] / \delta\lambda \\ &= -\bar{g}^{\mu\nu} B_\mu \left(F_\nu + \frac{a}{2} B_\nu \right) - i\bar{C}_\mu \bar{g}^{\mu\rho} \Delta_{\rho\nu}^{(gh)} C^\nu. \end{aligned} \quad (3.7)$$

The full form of the FP ghost term can be found in [28] and is very complicated, but in the single-metric approximation, it is much simpler and is given by [32]

$$\Delta_{\mu\nu}^{(gh)} \equiv \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} \bar{\nabla}_\rho \bar{\nabla}_\sigma + \left(1 - 2\frac{b+1}{d} \right) \bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{R}_{\mu\nu}. \quad (3.8)$$

What is important is that under the rescaling transformation (3.1) and (3.2), we have

$$\delta\sqrt{\bar{g}} = d\epsilon\sqrt{\bar{g}}, \quad \delta F_\mu = 0, \quad \delta(\bar{g}^{\mu\rho} \Delta_{\rho\nu}^{(gh)}) = -2\epsilon\bar{g}^{\mu\rho} \Delta_{\rho\nu}^{(gh)}. \quad (3.9)$$

Here we again encounter the same problem with the linear split unless $d = 2$ or we take the Landau gauge. In this gauge, we can just assign the rescaling properties as

$$\delta B_\mu = (2-d)B_\mu, \quad \delta \bar{C}_\mu = (2-d)\bar{C}_\mu, \quad \delta C^\mu = 0, \quad (3.10)$$

so that the gauge-fixing and FP terms are invariant. We again emphasize that we do not have to introduce higher-derivative gauge fixing if we just take the Landau gauge.

For the cutoff terms, we adopt a cutoff that is similar to that used by Percacci and Vacca [28] but suitably modified in the ghost part. Thus we consider

$$\begin{aligned}\Delta S_k(h_{\mu\nu}^T, \bar{g}_{\mu\nu}) &= \frac{1}{2} \int \sqrt{\bar{g}} [h_{\mu\nu}^T \bar{g}^{\mu\rho} \bar{g}^{\nu\rho} R_k^T(\bar{\Delta}) h_{\rho\sigma}^T + h R_k(\bar{\Delta}) h], \\ \Delta S_k^{gh}(\bar{C}_\mu, C^\mu, \bar{g}_{\mu\nu}) &= -i \int \sqrt{\bar{g}} \bar{C}_\mu R_k^{gh}(\bar{\Delta}) C^\mu,\end{aligned}\tag{3.11}$$

where we choose

$$R_k^T(\bar{\Delta}) = ck^d r(y), \quad R_k(\bar{\Delta}) = c_0 k^d r(y), \quad R_k^{gh}(\bar{\Delta}) = c_{gh} k^2 r(y).\tag{3.12}$$

Denoting $t = \ln k$, we then have

$$\delta R_k^T = \epsilon[\partial_t R_k^T - d R_k^T], \quad \delta R_k = \epsilon[\partial_t R_k - d R_k], \quad \delta R_k^{gh} = \epsilon[\partial_t R_k^T - 2 R_k^{gh}],\tag{3.13}$$

for the above cutoffs, and

$$\begin{aligned}\delta \Delta S_k &= \frac{\epsilon}{2} \int \sqrt{\bar{g}} [h_{\mu\nu}^T \bar{g}^{\mu\rho} \bar{g}^{\nu\rho} \partial_t R_k^T(\bar{\Delta}) h_{\rho\sigma}^T + h \partial_t R_k(\bar{\Delta}) h - 4 d R_k(\bar{\Delta}) h], \\ \delta \Delta S_k^{gh} &= -i \epsilon \int \sqrt{\bar{g}} \bar{C}_\mu \partial_t R_k^{gh}(\bar{\Delta}) C^\mu.\end{aligned}\tag{3.14}$$

Repeating the same manipulations as before, we find the mWI

$$\begin{aligned}& 2 \int \bar{g}_{\mu\nu} \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} + 2 \int \langle h_{\mu\nu}^T \rangle \frac{\delta \Gamma_k}{\delta \langle h_{\mu\nu}^T \rangle} - 2d \int \frac{\delta \Gamma_k}{\delta \langle h \rangle} \\ &= \frac{1}{2} \text{Tr} \left\{ \left(\frac{\delta^2 \Gamma_k}{\delta h^T \delta h^T} + R_k^T \right)^{-1} \partial_t R_k^T \right\} + \frac{1}{2} \text{Tr} \left\{ \left(\frac{\delta^2 \Gamma_k}{\delta h \delta h} + R_k \right)^{-1} \partial_t R_k \right\} \\ &\quad - \text{Tr} \left\{ \left(\frac{\delta^2 \Gamma_k}{\delta \bar{C} \delta C} + R_k^{gh} \right)^{-1} \partial_t R_k^{gh} \right\}.\end{aligned}\tag{3.15}$$

The main difference comes from the fact that here the traceless mode $h_{\mu\nu}^T$ transforms homogeneously but the trace part h inhomogeneously (see (3.3)) in contrast to the linear split, where $\delta h_{\mu\nu}^T = 0$ and $\delta h = -2\epsilon(h + d)$.

The right-hand side is identical to the right-hand side of the exact RG equation. We thus get

$$\int \left[2 \bar{g}_{\mu\nu} \frac{\delta \Gamma_k}{\delta \bar{g}_{\mu\nu}} + 2 \int \langle h_{\mu\nu}^T \rangle \frac{\delta \Gamma_k}{\delta \langle h_{\mu\nu}^T \rangle} - 2d \int \frac{\delta \Gamma_k}{\delta \langle h \rangle} \right] - k \frac{d \Gamma_k}{dk} = 0.\tag{3.16}$$

In the standard single-metric approximation [4] but slightly extended here, we keep only the dependence on the constant part \bar{h} . We then find that the solution to the mWI is given by

$$\Gamma = \hat{\Gamma}_{\hat{k}}[\hat{g}_{\mu\nu}],\tag{3.17}$$

where \hat{k} and $\hat{g}_{\mu\nu}$ are defined as

$$\hat{k} = e^{-\bar{h}/(2d)} k, \quad \hat{g}_{\mu\nu} = e^{\bar{h}/d} \bar{g}_{\mu\nu}, \quad \hat{h}_{\mu\nu}^T = e^{\bar{h}/d} h_{\mu\nu}^T.\tag{3.18}$$

We again see that the solution to the FRGE is written in terms of scale-independent variables and the coarse-graining problem may be resolved.

3.2 Gauge-fixing term and ghost

Here we discuss in more detail the contribution of the gauge-fixing term and FP ghost. For this purpose, it is more convenient to use the York type decomposition

$$h_{\mu\nu} = h_{\mu\nu}^{TT} + \bar{\nabla}_\mu \rho_\nu + \bar{\nabla}_\nu \rho_\mu - \frac{2}{d} \bar{g}_{\mu\nu} \bar{\nabla}^\alpha \rho_\alpha + \frac{1}{d} \bar{g}_{\mu\nu} h, \quad (3.19)$$

with

$$\bar{\nabla}^\nu h_{\mu\nu}^{TT} = \bar{g}^{\mu\nu} h_{\mu\nu}^{TT} = 0. \quad (3.20)$$

We then find [14]

$$F_\mu = \left[\bar{g}_{\mu\nu} \bar{\nabla}^2 + \frac{d-2}{2d} (\bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{\nabla}_\nu \bar{\nabla}_\mu) + \frac{d+2}{2} \bar{R}_{\mu\nu} \right] \rho^\nu - \frac{b}{d} \nabla_\nu h, \quad (3.21)$$

and the Jacobian from this must be taken into account. On the other hand, the ghost kinetic term is given by

$$\Delta_{\mu\nu}^{(gh)} = \bar{g}_{\mu\nu} \bar{\nabla}^2 + \frac{d-2-2b}{2d} (\bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{\nabla}_\nu \bar{\nabla}_\mu) + \frac{d+2+2b}{2} \bar{R}_{\mu\nu}. \quad (3.22)$$

We see that if we set $b = 0$, these contributions would cancel each other [14]. For this, it is necessary to use the same cutoff as the ghost term. In addition, in making the field transformation into ρ_μ in (3.19), we get the Jacobian $\text{Det}(\Delta - \frac{\bar{R}}{d})^{1/2}$ from that, and it should be taken into account with a similar cutoff as the FP ghost. This is the spin-1 contribution discussed in [17, 18].

4 Explicit example of the scale-independent flow equation for $f(R)$ gravity in $d = 4$

To get some idea of how the FRGE looks in this setting, let us consider an explicit example of the theory known as $f(R)$ gravity whose action is

$$S = \int d^d x \sqrt{-g} f(R). \quad (4.1)$$

We study this system with the exponential parametrization (1.3). Since the preceding discussions show that the scale-invariant FRGE may be obtained just by replacing the metric and momentum cutoff by scale-independent variables, we just check what FRGE is obtained with our gauge-fixing and cutoff scheme.

Following the standard procedure, we find the only difference from the known result is in the cutoff. We then arrive at the FRGE [17, 18]

$$\begin{aligned} \dot{\Gamma}_k = & \frac{1}{2} \text{Tr}_{(2)} \left[\frac{\dot{R}_k^T(\Delta)}{f'(\bar{R}) \left(\Delta + \alpha \bar{R} + \frac{2}{d(d-1)} \bar{R} \right) + R_k^T(\Delta)} \right] - \frac{1}{2} \text{Tr}_{(1)} \left[\frac{\dot{R}_k^{gh}(\Delta)}{\Delta + \gamma \bar{R} - \frac{1}{d} \bar{R} + R_k^{gh}(\Delta)} \right] \\ & + \frac{1}{2} \text{Tr}_{(0)} \left[\frac{\dot{R}_k(\Delta)}{f''(\bar{R}) \left(\Delta + \beta \bar{R} - \frac{1}{d-1} \bar{R} \right) + \frac{d-2}{2(d-1)} f'(\bar{R}) + R_k(\Delta)} \right], \end{aligned} \quad (4.2)$$

where the dot denotes the derivative with respect to the RG time $t = \log k/k_0$ (with k_0 an arbitrary reference scale) and $\Delta = -\nabla^2$ is the Laplacian. The subscripts on the traces represent contributions from different spin sectors: (2) denotes a trace over transverse-traceless symmetric tensor modes, (1) a trace over transverse-vector modes, and (0) a trace over scalar modes. Here α , β , and γ are free parameters, the choice of which corresponds to the choice of RG schemes along with the choice of the function $R_k(z)$. We note that the traces can in principle be evaluated for both negative and positive curvatures and in any dimension d . We give the necessary formulas in Appendix A.

Evaluation of the traces is done as follows: First, for some differential operator z , consider

$$\text{Tr}_{(j)}[W(z)] = \int_0^\infty ds \tilde{W}(s) \text{Tr}_{(j)}[e^{-sz}], \quad (4.3)$$

for the spin- j sector, where $\tilde{W}(s)$ is the inverse Laplace transform of $W(z)$:

$$W(z) = \int_0^\infty ds e^{-zs} \tilde{W}(s). \quad (4.4)$$

Using the heat kernel expansion

$$\text{Tr}_{(j)}[e^{-sz}] = \frac{1}{(4\pi s)^{d/2}} \int_{S^d} d^d x \sqrt{g} \sum_{n \geq 0} b_{2n}^{(j)} s^n \bar{R}^n, \quad (4.5)$$

in (4.3), we obtain

$$\text{Tr}_{(j)}[W(z)] = \frac{1}{(4\pi)^{d/2}} \int_{S^d} d^d x \sqrt{g} \sum_{n \geq 0} b_{2n}^{(j)} Q_{d/2-n}[W] \bar{R}^n, \quad (4.6)$$

where

$$Q_m[W] = \frac{1}{\Gamma(m)} \int_0^\infty dz z^{m-1} W[z]. \quad (4.7)$$

We choose the optimized cutoff profile [35] $r(y) = (1-y)\theta(1-y)$, where θ is the Heaviside distribution. For the contribution of the spin-2 modes in (4.2), we find

$$Q_m[W]_{(2)} = \frac{1}{\Gamma(m)} \int_0^\infty dz z^{m-1} \frac{(d-2)ck^{d-2}(k^2-z) + 2ck^d}{f'(\bar{R}) \left(z + \alpha \bar{R} + \frac{2}{d(d-1)} \bar{R} \right) + ck^{d-2}(k^2-z)} \theta(k^2-z). \quad (4.8)$$

We use the dimensionless quantities $r = \bar{R}k^{-2}$, $\varphi(r) = k^{-d}f(\bar{R})$, $\dot{f}(\bar{R}) = k^d[d\varphi(r) - 2r\varphi'(r) + \dot{\varphi}(r)]$, $f'(\bar{R}) = k^{d-2}\varphi'(r)$, and $f''(\bar{R}) = k^{d-4}\varphi''(r)$, and define $\tilde{\alpha} = \alpha + 2/d(d-1)$.

From now on, we set $d = 4$. We then obtain

$$\begin{aligned} Q_m[W]_{(2)} &= \frac{2ck^{2m}}{\Gamma(m)} \int_0^1 dy y^{m-1} \frac{2-y}{\varphi'(r)(y + \tilde{\alpha}r) + c(1-y)} \\ &= \frac{2ck^{2m}}{\Gamma(m+2)[c + \tilde{\alpha}r\varphi'(r)]^2} \left[2(m+1)[c + \tilde{\alpha}r\varphi'(r)] \right. \\ &\quad \left. + m[c - \varphi'(r)(2 + \tilde{\alpha}r)] {}_2F_1[1, 1+m, 2+m, \frac{c - \varphi'(r)}{c + \tilde{\alpha}r\varphi'(r)}] \right], \end{aligned} \quad (4.9)$$

where ${}_2F_1[a, b, c, z]$ is the hypergeometric function. More explicitly

$$\begin{aligned}
Q_2[W]_{(2)} &= -\frac{ck^4}{(c - \varphi'(r))^3} \left[\{c - \varphi'(r)\} \{c - (3 + 2\tilde{\alpha}r)\varphi'(r)\} \right. \\
&\quad \left. + 2[c + \tilde{\alpha}r\varphi'(r)][c - (2 + \tilde{\alpha}r)\varphi'(r)] \log \left\{ \frac{(1 + \tilde{\alpha}r)\varphi'(r)}{c + \tilde{\alpha}r\varphi'(r)} \right\} \right], \\
Q_1[W]_{(2)} &= \frac{2ck^2}{(c - \varphi'(r))^2} \left[c - \varphi'(r) - [c - (2 + \tilde{\alpha}r)\varphi'(r)] \log \left\{ \frac{(1 + \tilde{\alpha}r)\varphi'(r)}{c + \tilde{\alpha}r\varphi'(r)} \right\} \right], \\
Q_0[W]_{(2)} &= \frac{4c}{c + \tilde{\alpha}r\varphi'(r)}, \quad Q_{-1}[W]_{(2)} = -\frac{2c[c - (2 + \tilde{\alpha}r)\varphi'(r)]}{k^2(c + \tilde{\alpha}r\varphi'(r))^2}.
\end{aligned} \tag{4.10}$$

Similarly, for spin 1 we find

$$Q_m[W]_{(1)} = \frac{2c^{gh}}{\Gamma(m)} \int_0^\infty dz z^{m-1} \frac{k^2}{z + \tilde{\gamma}\bar{R} + c^{gh}(k^2 - z)} \theta(k^2 - z), \tag{4.11}$$

where we have defined $\tilde{\gamma} = \gamma - \frac{1}{4}$. The relevant results are

$$\begin{aligned}
Q_2[W]_{(1)} &= \frac{2c^{gh}k^4}{(c^{gh} - 1)^2} \left[1 - c^{gh} - (c^{gh} + \tilde{\gamma}r) \log \left\{ \frac{1 + \tilde{\gamma}r}{c^{gh} + \tilde{\gamma}r} \right\} \right], \\
Q_1[W]_{(1)} &= \frac{2c^{gh}k^2}{1 - c^{gh}} \log \left\{ \frac{1 + \tilde{\gamma}r}{c^{gh} + \tilde{\gamma}r} \right\}, \\
Q_0[W]_{(1)} &= \frac{2c^{gh}}{c^{gh} + \tilde{\gamma}r},
\end{aligned} \tag{4.12}$$

while for spin 0 we have

$$Q_m[W]_{(0)} = \frac{2c_0k^{2m}}{\Gamma(m)} \int_0^1 dy y^{m-1} \frac{2 - y}{\varphi''(r) \left(y + \tilde{\beta}r \right) + \frac{1}{3}\varphi'(r) + c_0(1 - y)}, \tag{4.13}$$

with $\tilde{\beta} = \beta - \frac{1}{3}$. We find

$$\begin{aligned}
Q_2[W]_{(0)} &= -\frac{c_0k^4}{9[c_0 - \varphi''(r)]^3} \left[3\{c_0 - \varphi''(r)\} \{3c_0 - 2\varphi'(r) - 3(3 + 2\tilde{\beta}r)\varphi''(r)\} \right. \\
&\quad + 2\left[9\{c_0 + \tilde{\beta}r\varphi''(r)\} \{c - (2 + \tilde{\beta}r)\varphi''(r)\} - \varphi'(r)^2 \right. \\
&\quad \left. \left. - 6(1 + \tilde{\beta}r)\varphi'(r)\varphi''(r) \right] \log \left\{ \frac{\varphi'(r) + 3(1 + \tilde{\beta}r)\varphi''(r)}{3c_0 + \varphi'(r) + 3\tilde{\beta}r\varphi''(r)} \right\} \right], \\
Q_1[W]_{(0)} &= \frac{2c_0k^2}{3[c_0 - \varphi''(r)]^2} \left[3c_0 - 3\varphi''(r) \right. \\
&\quad \left. - \{3c_0 - \varphi'(r) - 3(2 + \tilde{\beta}r)\varphi''(r)\} \log \left\{ \frac{\varphi'(r) + 3(1 + \tilde{\beta}r)\varphi''(r)}{3c_0 + \varphi'(r) + 3\tilde{\beta}r\varphi''(r)} \right\} \right], \\
Q_0[W]_{(0)} &= \frac{12c_0}{3c_0 + \varphi'(r) + 3\tilde{\beta}r\varphi''(r)}, \\
Q_{-1}[W]_{(0)} &= \frac{6c_0 \left[-3c_0 + \varphi'(r) + 3(2 + \tilde{\beta}r)\varphi''(r) \right]}{k^2[3c_0 + \varphi'(r) + 3\tilde{\beta}r\varphi''(r)]^2},
\end{aligned} \tag{4.14}$$

The heat kernel coefficients b_{2n} for Δ acting on spin-2, 1, and 0 are given in [17, 18] for our case and summarized in Appendix A. Substituting these heat kernel coefficients and Eqs. (4.9), (4.11), and (4.13) in (4.2), we obtain

$$\begin{aligned}
2(4\pi)^2(\dot{\varphi} - 2r\varphi' + 4\varphi) = & b_0^{(0)}q_2^{(0)} - b_0^{(1)}q_2^{(1)} + b_0^{(2)}q_2^{(2)} \\
& + \left(b_2^{(0)}q_1^{(0)} - b_2^{(1)}q_1^{(1)} + b_2^{(2)}q_1^{(2)}\right)r \\
& + \left(b_4^{(0)}q_0^{(0)} - b_4^{(1)}q_0^{(2)} + b_4^{(2)}q_0^{(2)}\right)r^2 \\
& + \left(b_6^{(0)}q_{-1}^{(0)} + b_6^{(2)}q_{-1}^{(2)}\right)r^3,
\end{aligned} \tag{4.15}$$

where we have defined $q_m^{(i)} \equiv Q_m[W]_{(i)}/k^{2m}$, which are k -independent. The scale-independent solution is obtained from the solution of this flow equation just by using the scale-independent variables (3.18).

We thus find the flow equation in terms of the scale-independent variables. The explicit example looks rather complicated with logarithms. The next problem would be to try to see what solutions it allows, but a full analysis is beyond the scope of this paper.

5 Conclusions

In this paper we have been able to extend the scale-independent FRGE formulated in six dimensions [21] to arbitrary dimensions. The crucial point in achieving this is the recognition of the necessity of the Landau gauge and a change of the cutoff scheme. It has been pointed out that this can be also done if one uses the exponential split of the metric, higher-derivative gauge fixing and pure cutoff [28]. However, we believe that the first condition, the exponential split, may not be necessary as we have shown. It is certainly true that if we use higher-derivative gauge fixing, it is possible to realize the scale independence for arbitrary gauge fixing parameters. However the use of higher-order gauge fixing is unusual, and it is nice to see that it is possible to formulate it with the often-used gauge fixing. We have seen that it is indeed possible if we adopt the Landau gauge. Thus the first two conditions, i.e. exponential split of the metric and higher-derivative gauge fixing are not required, but the third is essential.

Even though this is true, the exponential split has various advantages like avoidance of unphysical singularities [17, 18, 19, 30, 31, 32]. So we have studied the problem in both linear and exponential splits, and have shown that it is possible to formulate the background independence in this approach. These are not unique choices realizing the background independence, but most commonly used parametrizations.

To get some idea of what the resulting FRGE looks like, we have also given it for the case of $f(R)$ gravity. Because we have to use the pure cutoff scheme, the resulting FRGE becomes quite complicated. The full analysis of its solutions requires quite a lot of work, and is left for future study. It would be very interesting to give solutions to this equation.

Acknowledgments

I am grateful for numerous valuable discussions with Tim Morris and Roberto Percacci. This work was supported in part by the Grant-in-Aid for Scientific Research Fund of the Japan Society for the Promotion of Science (C) No. 16K05331.

A Heat kernel coefficients on the d -sphere

The heat kernel coefficients can be found by summing over eigenvalues $\lambda_\ell(d, s)$ of the operator Δ weighted by their multiplicity $M_\ell(d, s)$

$$\text{Tr}_{(s)}[e^{-\sigma(\Delta+E_{(s)})}] = \sum_{\ell} M_\ell(d, s) e^{-\sigma(\lambda_\ell(d, s)+E_{(s)})}. \quad (\text{A.1})$$

For general d , $\lambda_\ell(d, s)$ and $M_\ell(d, s)$ are summarized in Table 1.

Spin	Eigenvalue $\lambda_\ell(d, s)$	Multiplicity $M_\ell(d, s)$	
0	$\frac{\ell(\ell+d-1)}{d(d-1)} \bar{R}$	$\frac{(2\ell+d-1)(\ell+d-2)!}{\ell!(d-1)!}$	$\ell = 0, 1, \dots$
1	$\frac{\ell(\ell+d-1)-1}{d(d-1)} \bar{R}$	$\frac{\ell(\ell+d-1)(2\ell+d-1)(\ell+d-3)!}{(d-2)!(\ell+1)!}$	$\ell = 1, 2, \dots$
2	$\frac{\ell(\ell+d-1)-2}{d(d-1)} \bar{R}$	$\frac{(d+1)(d-2)(\ell+d)(\ell-1)(2\ell+d-1)(\ell+d-3)!}{2(d-1)!(\ell+1)!}$	$\ell = 2, 3, \dots$

Table 1: Eigenvalues and multiplicities of the Laplacian on the d -sphere.

We use the Euler-MacLaurin formula

$$\sum_{n=a}^b f(n) = \int_a^b f(x) dx + \frac{f(b) + f(a)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right). \quad (\text{A.2})$$

Here B_{2k} denotes the Bernoulli numbers, and the boundaries are $a = 2$ and $b = \infty$. Naively one would expect that $a = 0$ ($s = 0$), $a = 1$ ($s = 1$), $a = 2$ ($s = 2$), and $b = \infty$. However one has to leave out the mode $n = 1$ for the spin-1 field ξ_μ (Killing vectors) and for the field σ one has to leave out the modes $n = 0$ (constant) and $n = 1$ (related to the five conformal Killing vectors that are not Killing vectors), so the sum should start from $n = 2$.

For $d = 4$, the functions $f^{(s)}(x)$ entering into (A.2) are

$$f^{(0)}(x) = \frac{1}{6}(x+1)(x+2)(2x+3)e^{-\frac{1}{12}x(x+3)\bar{R}\sigma+\beta\bar{R}\sigma},$$

$$f^{(1)}(x) = \frac{1}{2}x(x+3)(2x+3)e^{-\frac{1}{12}\{x(x+3)-1\}\bar{R}\sigma+\gamma\bar{R}\sigma}, \quad (\text{A.3})$$

$$f^{(2)}(x) = \frac{5}{6}(x-1)(x+4)(2x+3)e^{-\frac{1}{12}\{x(x+3)-2\}\bar{R}\sigma+\alpha\bar{R}\sigma}. \quad (\text{A.4})$$

The integral parts in (A.2) are given by

$$\begin{aligned} \int_2^\infty dx f^{(0)}(x) &= \frac{1}{(4\pi\sigma)^2} \int_{S^d} d^d x \sqrt{g} (1 + \bar{R}\sigma) e^{-\frac{5\bar{R}\sigma}{6}+\beta\bar{R}\sigma}, \\ \int_2^\infty dx f^{(1)}(x) &= \frac{1}{(4\pi\sigma)^2} \int_{S^d} d^d x \sqrt{g} \left(3 + \frac{5}{2}\bar{R}\sigma \right) e^{-\frac{3\bar{R}\sigma}{4}+\gamma\bar{R}\sigma}, \\ \int_2^\infty dx f^{(2)}(x) &= \frac{1}{(4\pi\sigma)^2} \int_{S^d} d^d x \sqrt{g} \left(5 + \frac{5}{2}\bar{R}\sigma \right) e^{-\frac{2\bar{R}\sigma}{3}+\alpha\bar{R}\sigma}. \end{aligned} \quad (\text{A.5})$$

We find the coefficients for $d = 4$ given in Table 2 [17, 18].

Spin	b_0	b_2	b_4	b_6
0	1	$\frac{1}{6} + \beta$	$\frac{-511+360\beta+1080\beta^2}{2160}$	$\frac{19085-64386\beta+22680\beta^2+45360\beta^3}{272160}$
1	3	$\frac{1}{4} + 3\gamma$	$\frac{-607+360\gamma+2160\gamma^2}{1440}$	$\frac{37259-152964\gamma+45360\gamma^2+181440\gamma^3}{362880}$
2	5	$-\frac{5}{6} + 5\alpha$	$\frac{-1-360\alpha+1080\alpha^2}{432}$	$\frac{311-126\alpha-22680\alpha^2+45360\alpha^3}{54432}$

Table 2: Heat kernel coefficients for $d = 4$.

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